ON STABILITY OF ROSSBY-HAURWITZ WAVES AND VERKELEY'S MODONS IN THE BAROTROPIC ATMOSPHERE

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ABSTRACT

Stability of Verkeley's modons and Rossby-Haurwitz (R-H) waves of subspace H1 Hn is analyzed within the vorticity equation of an ideal incompressible fluid on a rotating sphere. Here Hn is the eigen subspace of the spherical Laplace operator corresponding to the eigenvalue $n(n+1)$. Conservation laws and independent sets of any perturbations of R-H waves and small perturbations of Verkley's modon are found.

It is proved that any super-rotation flow is Lyappunov stable but any dipole modon or non-zonal R-H wave of H1 Hn where $n \geq 2$ are Lyapunov unstable because of the algebraic growth of perturbations caused by asynchronous oscillations of waves.

R-H Waves and Verkely's modons on a sphere

Dynamics of the invicid and unforced barotropic atmosphere on a rotating unit sphere S is decribed by the nondimensional vorticity equation.

$$\frac{\partial}{\partial t} \Delta \psi + J(\psi, \Delta \psi + 2\mu) = 0$$

Here $\psi(t, \lambda, \mu)$ is the stream function, $\mu = \sin\varphi, \varphi$ and λ are the latitude and longitude of a point (λ, μ) on S, $\Delta\psi$ and $\Delta\psi + 2\mu$ are the relative and absolute vorticity respectively, Δ is the Laplace operator on S and $J(\psi, h)$ is the Jacobian.

The Rossby-Haurwitz (R-H) wave

$$f(t, \lambda, \mu) = -\omega\mu + \sum_{m=-n}^{n} f_m y_n^m (\lambda - c_n t, \mu) \quad (2)$$

$(n \geq 1)$ is the exact solution of Eq. (1) if its speed satisfies

$$C_n = \omega - \frac{2(\omega + 1)}{\chi_n} \quad (3)$$

The first term in (2) is the solid body rotation with the angular velocity ω and the second term being a linear combination of the spherical harmonics $Y_n^m(\lambda, \mu)$ is the homogeneous spherical polynomial of degree n.

The Verkley (1984, 1987) modon

$$\psi(\lambda', \mu') = R(\mu')\cos\lambda' + G(\mu') \quad (4)$$

is another type of exact periodic solution to (1) provided the pole $N' = (\lambda_o, \mu_o)$ of the primed coordinate system (λ', μ') moves along a latitude circle $\mu_o = $ Const with the constant velocity.

$$C_n = \omega_o - \frac{2(\omega_o + 1)}{\chi_\sigma} \quad (5)$$

The harmonics $P_\sigma^m(\mu)\cos m\lambda$ and $P_\alpha^m(m)\cos m\lambda$ $(m = 0,1)$ are used for constructing the modon (4) in the outer and inner regions $S_o(\mu' < \mu_a)$ and $S_i(\mu' > \mu_a)$ respectively. In this work ω_o is the super-rotation velocity in $S_o, \chi_\sigma = \sigma(\sigma+1)$. and $\chi_\alpha = \alpha(\alpha+1)$. The degree α is real ($\alpha \geq 2$) and $\chi_\alpha > 0$. The degree α may be real or complex. In last case $\alpha = -\frac{1}{2} + ik$ where $k > 0$, and the modon is localized mainly in a small neighborhood of the region S_i, besides, $\chi_\alpha = -(k^2 + \frac{1}{4}) < 0$. In particular, if $\alpha = \sigma$, then the modon (4) is the R-H wave.

If $|\mu_o| = 1$ then in the unprimed coordinates (4) is a zonal flow: $\psi(\mu) = G(\mu)$ (a monopole modon, Verkeley (1984)). Modon is called dipole if $|\mu_o| \neq 1$. The functions $R(\mu)$ and $G(\mu)$ are smooth except the point $\mu = \mu_a$ where they have continuous derivatives only up to the second order.

In this work we study the stability of the solutions (2) and (4). As any conservative system, the system described by (1) is structurally unstable and hence only exact analytical methods should be used to obtain correct stability results.

Conservation law and invariant sets of arbitrary pertubations of the R-W wave.

It is shown in Skiba (1989) that any perturbation $\psi'(\lambda, \mu, t)$ of the R-H wave (2), (3) evolves such that its kinetic energy $K(t)$ and enstrophy $\gamma(t)$ defined as

$$K(t) = \frac{1}{2}\int_s |\nabla \psi'(\lambda, \mu, t)|^2 dS \text{ and } \gamma(t) = \frac{1}{2}\int_s |\Delta \mu'(\lambda, \mu, t)|^2 dS$$

decrease, conserve or increase simultaneously due to the law

$$\frac{d}{dt}\gamma(t) = \chi_n \frac{d}{dt}K(t) \quad (6)$$

Thus, although the enstrophy $\gamma(t)$ is more strong characteristics than the energy $K(t)$, nevertheless, in the stability study of the R-H wave (2), the perturbation energy $K(t)$ can be used instead of $\gamma(t)$ without loss of generality. By (6) we have that

$$\frac{d}{dt} \{ [\chi(\psi') - \chi_n] \; k(t) \} = 0 \qquad (7)$$

for any perturbations ψ' of the wave (2). Here $\chi_n = n(n+1)$ and

$$\chi(\psi') = \gamma(t) / K(t) \qquad (8)$$

is the mean spectral number of ψ'. By (7) there is a hyperbolic dependence between the energy $K(t)$ and characteristics $\chi(\psi') - \chi_n$ of the spectral composition of the wave (2) perturbation.

Due to (7), all possible perturbations of the R-H wave (2), (3) can be divided into three invariant (independent) sets:

$$M_+^n = \{ \psi' : \chi(\psi') > \chi_n \} , \; M_o^n = \{ \psi' : \chi(\psi') = \chi_n \} ,$$

$$M_-^n = \{ \psi' : \chi(\psi') < \chi_n \} \qquad (9)$$

Thus, any perturbation (independently of its amplitude) initially belongign to one of the sets (9) will never leave it. It allows to analyze the R-H wave stability separately in each of the sets (9). Note that the R-H wave (2) is Lyapunov stable to any perturbation of the set Hn of homogeneous shperical polynomials of degree n which is one more invariant set belonging to M_o^n (Skiba, 1989). Further, by (6) and (8),

$$\frac{d}{dt} \chi(\psi') = \frac{d}{dt} \{ \gamma(t)/K(t) = \frac{1}{k(t)} \{ \chi_n - \chi(\psi') \} \frac{d}{dt} K(t) \qquad (10)$$

and hence, the energy cascade of growing perturbations of the R-H wave has the opposite directions in the sets M_-^n and M_+^n: the derivative $\frac{d}{dt} \chi(\psi')$ is positive in M_-^n and negative in M_+^n. Besides, due to (7), the closer the mean spectral number $\chi(\psi')$ to χ_n, the larger is the perturbation energy. Note that, by (9), the energy cascade to the smaller scales is bounded for perturbations of M_-^n.

Lyapunov instability of the non-zonal R-H wave

As is well known, the zero solution of (1) us Lyapunov stable (Lyapunov, 1966) in the norm

$$\| \psi' \|_* = (e_1 K + e_2 \gamma)^{½} \qquad (11)$$

with respect to arbitrary perturbations because the kinetic energy and enstrophy of any solutions to (1) are constant with time (Fjortoft, 1953). It is shown that any super-rotation flow (or any homogeneous spherical polynomial of degree 1) is also Lyapunov stable in the norm (11),

besides, both the kinetic energy and enstrophy of any perturbation of such a flow are conserved in time (Skiba, 1989). But stability porperties of the R-H wave (2) are quite different if $n \geq 2$. Indeed, if $n \geq 2$ then any non-zonal R-H wave (2), (3) is Lyapunov unstable in the norm (11) to some perturbations of M_-^n (Skiba, 1991, 1992).

We now consider a simple example so as to bring to light the mechanism of Lyapunov instability of the non-zonal R-H waves. Let

$$f(t,\lambda,\mu) = F Q_n^m(\mu) \cos m(\lambda - c_n t) \qquad (12)$$

be a basic solution where $Q_n^m(\mu)$ is the normalized associated Legendre funtion and F is the real amplitude ($m \geq 1$). The kinetic energy of the wave (12) is $K_f = \pi \chi_n F^2$ where $\chi_n = n(n+1)$. Let $\varepsilon^2 = 4 K_f$ and $\delta > o$ be as small as we like. As another solution to (1) we take

$$\psi(t,\lambda,\mu) = -\frac{\delta}{2} y_1^o(\mu) + F Q_n^m(\mu) \cos m(\lambda - \hat{C}_n t) \quad (13)$$

Then $\hat{C}_n - C_n = \delta (\chi_n - 2) / 2\chi_n$ is a very small number. The corresponding perturbation $\psi' = \psi - f$ of the wave (12) is

$$\psi'(t,\lambda,\mu) = -\frac{\delta}{2} y_1^o(\mu) + A(t) Q_n^m(\mu) \sin m (\lambda - \frac{\hat{c}_n+c_n}{2} t) \quad (14)$$

where

$$A(t) = 2 F \sin m \frac{\delta (\chi_n - 2)}{4 \chi_n} t \qquad (15)$$

Then the kinetic energy $K(t)$ of the perturbation (14) is

$$K(t) = \frac{\delta^2}{4} + \pi \chi_n A^2(t) \qquad (16)$$

Let us take $\| \psi' \|_* = K^{½}$ as the norm (11). Then at initial moment $t_o = 0$ the amplitude $A(0) = 0$ and $K(0) = \frac{\delta^2}{4}$ and hence, $\| \psi'(0) \|_* < \delta$. However, at $t = \tau_j = \frac{2(1+2j)\pi\chi_n}{m \delta(\chi_n - 2)}$ where $j = 0,1,2,...$ we have $A(\tau_j) = 2 F$ and

$$K(\tau_j) = \frac{\delta^2}{4} + 4 K_f = \frac{\delta^2}{4} + \varepsilon^2 > \varepsilon^2 \qquad (17)$$

i.e., $\| \psi'(0) \|_* > \varepsilon$. Hence the R-H wave (12) is Lyapunov unstable in the norm (11) (Skiba, 1992).

By (15), the perturbation amplitude, $A(t)$ varies in direct proportion to the amplitude F of the basic solution. Besides, the maximum of $A(t)$ is twice that of the basic wave (12) and is independent of δ. Hence, if the basic wave amplitude F is llarge then the growth of the perturbation amplitude $A(t)$ is considerable and evident independently of initial distance δ between two R-H waves (12) and (13). And if F is small then the growth of $A(t)$ is not so visible. In addition, since δ is very small, the growth of the amplitude $A(t)$ is slow. Note that the Lyapunov instability of the non-zonal R-H wave has nothing in common with the orbital (or

Lagrange) instability (Skiba, 1993). The reason for the Lyapunov instability is the non-zero shift $\hat{C}_n - C_n$ of the velocities of two R-H waves whose paths are close to each other in phase space, i.w., the asynchronous oscillations of the waves.

Finally, although the choice of the Lyapunov functional is essential in the Lyapunov stability study, the formulas (14) and (15) give conclusive evidence of the growth in amplitude of the perturbation, and hence, of the instability.

The law (6) can be written as

$$\sum_{k=n+1}^{\infty} \chi_k(\chi_k-\chi_n) \sum_{m=-k}^{k} |\psi'^m_k(t)|^2 - \sum_{k=1}^{n-1} \chi_k(\chi_n-\chi_k) \sum_{m=-k}^{k} |\psi'^m_k(t)|^2 = L_o \quad (18)$$

where $\psi'Km(t)$ is the Fourier coefficient of the perturation ψ' and the constant Lo is determined by initial perturbation. By (18) the both sums in the left part of (18) may grow only simultaneously. In Petroni et al (1987) such growth is considered as a necesary condition for the Rossby wave instability. But, the perturbation (14) shows that the Lyapunov instability is possible even if two sums in (18) remain constant in time. It conforms the point of Sheperd (1988) in his dispute on the double cascade mechanism with Petroni et al (1987,1989).

Dynamics of perturbations of Verkeley's modon

It is shown in Skiba (1993) that there exists the conservation law

$$\frac{d}{dt}\{K - \chi_\alpha^{-1}\gamma_i - \chi_\sigma^{-1}\gamma_o\} = 0 \quad (19)$$

for small perturbations of any stationary modon (4) on the sphere. Here γ_i and γ_o are the parts of the perturbation enstrophy $\gamma(t)$ containing in the regions S_i and S_o of the modon. The law is analogous to that obtained by Laedke and Spatschek (1986) for the beta-plane modon. Denote

$$\chi_i(t) = \gamma_i(t) / K(t) \ , \ \chi_o(t) = \gamma_o(t) / K(t)$$

The $\chi(t) = \chi_i(t) + \chi_o(t)$ is the perturbation mean spectral number and, due to (19) all small perturbations of the stationary modon are divided into three independent sets.

$$M_- = \{\psi': p(\psi') < 1\}, M_o = \{\psi'; p(\psi') = 1\}$$

$$M_+ = \{\psi': p(\psi') > 1\} \quad (20)$$

where the number $p(\psi') = \chi_\alpha^{-1}\chi_i(t) + \chi_\sigma^{-1}\chi_o(t)$ characterizes the spectral composition of the perturbation ψ'. We have

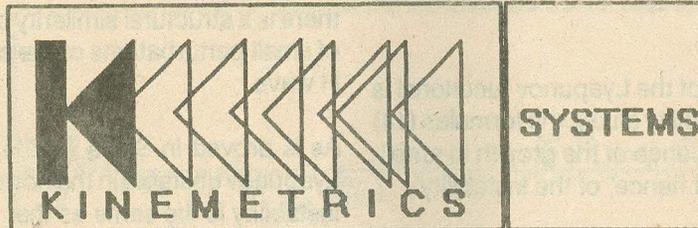$$\frac{d}{dt} p(\psi') = \frac{1}{k(t)} \{1 - p(\psi')\} \frac{d}{dt} k(t) \quad (21)$$

If $\chi_\alpha = \chi_\sigma$ then (19) is identical to (10). If $\chi_\alpha \neq \chi_\sigma$ then (19) is valid only for small perturbations. By (21), the energy cascade of growing perturbations of the stationary moden has the opposite directions in the sets M_- and M_+. Thus there is a structural similarity between of the invariant sets of small perturbations of the stationary modon and the R-H wave.

As is proved in Skiba (1991), any dipolo modon (21) is Lyapunov unstable in the norm (11), besides, nature of the instability is the same as that of the R-H wave.
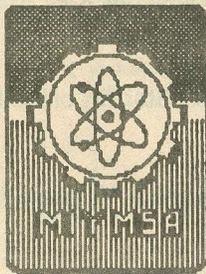
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